PolInSAR Coherence Set Theory and Application
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Abstract

In this paper we investigate coherence sets in polarimetric SAR interferometry. The coherence set is closely related to the numerical range (field of values). This paper aims to bring the numerical range to the attention of the radar research community. The mathematical understanding of the numerical range helps to understand theoretical and physical properties of coherence and coherency matrix. This paper presents a structured hierarchy of coherence sets and corresponding numerical ranges. Different applications which are related to numerical ranges are reviewed. An example of the inverse analysis of the coherence set is performed to obtain information about the coherency matrix. Finally, a generalization of the coherence set applications is presented.

1 Introduction

Polarimetric SAR Interferometry (POLInSAR) is a rapidly developing research field. It is already applied in many various areas, for instance, in physical parameter inversion, enhanced topography mapping, or change detection. The main observable in POLInSAR is the coherency matrix. From the coherency matrix we can compute the coherences for arbitrary polarizations. These coherences are complex scalars and are located inside the unitary circle of the complex plane. All possible coherences fill a continuous region in this complex plane and build up the POLInSAR coherence set for the respective coherency matrix.

The coherence set can be related with the numerical range (NR), also known as the field of values. This concept was first introduced to POLInSAR applications by Tabb and Flynn [4, 7], and later it was utilized by Colin [3]. The numerical range is a mathematical concept and has been widely researched. NR has many interesting properties and allows the direct investigation of the kernel matrix. Mathematical understanding of the NR helps to understand theoretical and physical properties of coherence and coherency matrix.

There are different approaches for utilization of the NR and the coherence set concept. Very common are optimization methods that try to find coherences, coherence phases, or corresponding polarizations inside the coherence set. They differ in the definition of the optimum and the amount of scattering mechanism space. The first developed optimization technique [1] maximizes the modulus of coherences. It is a general coherence optimization technique which considers different possible scattering mechanisms along the baseline. In [3] a coherence modulus optimization method is proposed, that is constrained to utilize only equal scattering mechanisms at the two baseline ends. Contrary to the previous modulus optimization techniques, the phase diversity [7] technique locates the phase extrema. Many other coherence optimization approaches exist.

In [6] the POLInSAR coherence set was examined from another viewpoint. This method approximates the coherence extension by a simple figure, i.e. an ellipse. Based on the derived parameters a decomposition and a classification is performed. Experiments have demonstrated that a high information content is present in the coherence region shape and distribution.

The goal of this paper is, as continuation of [3, 4, 7], to investigate and characterize the different types of coherence sets and numerical ranges. Also we will review the application of coherence sets and emphasize possible new application fields. Section 2 presents the fundamentals of the POLInSAR coherence matrix and coherence. In section 3, the concept of coherence sets is introduced. Relationships to numerical ranges are presented. In section 4, applications of the coherence sets are reviewed. Also, important properties of NR are shown with relation to the usage in POLInSAR. As an example application, an interpretation of a coherence set is shown in 4.2. The properties of the coherency matrix are predicted in the knowledge of the coherence set shape for the random volume over ground model. Finally, in 4.3 we will show that methods using approximated coherences can be modified for the use of non–approximated coherences.

2 Polarimetric SAR Interferometry

The usual representation of mono-static polarimetric interferometric data is the coherency matrix $\mathbf{T}_k$ which is built through multi–looking of coherent scattering vectors $\mathbf{k}$,
the Pauli matrix–basis:
\[
K_i = \frac{1}{\sqrt{2}} [S_{HHi} + S_{VVi}, S_{HHi} - S_{VVi}, 2S_{HVi}]^T
\]  
(1)
\[
T_6 = (Kk^T) = \begin{bmatrix} T_{11} & \Omega_{12} \\ \Omega_{12} & T_{22} \end{bmatrix}
\]  
(2)
where \(S_{HHi}, S_{VVi}\) and \(S_{HVi}\) are elements of two \((i = \{1, 2\})\) polarimetric scattering matrices. These scattering matrices are measured along a baseline with slightly different incidence angles and possibly at different times.

The definition of PolInSAR complex coherence was introduced in [1] as:
\[
\gamma = |\gamma|e^{i\phi} = \gamma_{(\omega_1, \omega_2)} = \frac{\omega_1^\dagger \Omega_{12} \omega_2}{\sqrt{\omega_1^\dagger T_{11} \omega_1} \sqrt{\omega_2^\dagger T_{22} \omega_2}}
\]  
(3)
where the normalized projection vectors \(\omega_1 (\omega_1^\dagger \omega_1 = 1)\) can be interpreted as scattering mechanisms.

An equivalent and easier to handle representation of the PolInSAR coherence can be given with the use of the modified normalized coherency matrix \(T_6\) and \(\Pi\):
\[
T_6 = \begin{bmatrix} I_3 & \Pi \\ \Pi^\dagger & I_3 \end{bmatrix} = RT_6 R,
\]
\[
R = \begin{bmatrix} T_{11}^\frac{1}{2} & 0 \\ 0 & T_{22}^\frac{1}{2} \end{bmatrix}
\]  
(4)
\[
\Pi = T_{11}^{-\frac{1}{2}} \Omega_{12} T_{22}^\frac{1}{2}
\]  
(5)
With substitutions \(w_i = \sqrt{\omega_i^\dagger T_{11} \omega_1}\) equation (3) becomes:
\[
\gamma = |\gamma|e^{i\phi} = \gamma_{(w_1, w_2)} = w_1^\dagger \Pi w_2, \quad w_1^\dagger w_2 = 1
\]  
(6)
At this point we want to outline the big amount of useful information as well as unwanted decorrelation in the coherence. This information is not easy to extract, which advocates for the fundamental study of the coherence behavior. In general, the modulus of the coherence corresponds to the correlation of the signal at the two ends of the baseline, whereas the phase is in the first place related to the topography. We present two simplified models of the coherence.

The absolute value of the complex coherence, \(|\gamma|\), can be modeled as:
\[
|\gamma| = |\gamma_{temp}| \cdot |\gamma_{vol}| \cdot |\gamma_{base}| \cdot |\gamma_{de}| \cdot |\gamma_{system}|
\]  
(7)
\(|\gamma_{temp}|, \gamma_{vol}, \gamma_{base}\) and \(\gamma_{de}\) correspond to temporal decorrelation, volume decorrelation, baseline decorrelation and Doppler centroid decorrelation, respectively. \(\gamma_{system}\) includes other noise contributions like thermal noise, coregistration errors, interpolation errors, etc.

The phase \(\phi\) of the complex coherence is the interferometric phase of the scattering mechanisms and is composed by:
\[
\phi = \phi_{topo} + \phi_{flat} + \Delta \phi_{prop} + \Delta \phi_{scat} + \Delta \phi_{SR} + \phi_{noise}
\]  
(8)
The contributions in (8) are topographic phase \(\phi_{topo}\), flat earth phase \(\phi_{flat}\), phase difference due to (ionospheric and atmospheric) propagation conditions \(\Delta \phi_{prop}\), phase differences due to change in scattering behavior \(\Delta \phi_{scat}\) and due to displacement or deformation of the scatterer \(\Delta \phi_{SR}\), as well as phase noise due to signal decorrelation \(\phi_{noise}\).

The ability to calculate the coherence for any polarization versus single polarization does not only improve the accuracy of information retrieval, but also permits new methods to extract additional parameters.

### 3 POLInSAR Coherence Set and the Numerical Range

#### 3.1 General Coherence Set

In order to derive another representation of POLInSAR data we build a coherence set \(\Gamma\) over the constrained scattering mechanism space \(\omega_1 \times \omega_2\). According to coherence definition in [1] and in equations (3), (6) we build the most general coherence set by:
\[
\Gamma = \{\gamma_{(\omega_1, \omega_2)} : \omega_1, \omega_2 \in \mathbb{C}^n, \omega_1^\dagger \omega_1 = \omega_2^\dagger \omega_2 = 1\}
\]
\[
= \{w_1^\dagger \Pi w_2 : w_1, w_2 \in \mathbb{C}^n, w_1^\dagger w_1 = w_2^\dagger w_2 = 1\}
\]  
(9)
This formulation is closely related to the numerical range. The numerical range \(W(A)\) (also known as the field of values) of a square matrix \(A \in \mathbb{C}^{n \times n}\) is a set of complex numbers over the Hermitian form:
\[
W(A) = \{x^\dagger Ax : x \in \mathbb{C}^n, x^\dagger x = 1\}
\]  
(10)
It contains useful information about the matrix and interesting properties, like e.g. the guaranteed convexity of the set. Replacing the Hermitian form \(x^\dagger Ax\) by the sesquilinear form \(y^\dagger Ax\) leads to the \(q\)-numerical range:
\[
W_q(A) = \{y^\dagger Ax : x, y \in \mathbb{C}^n, y^\dagger y = x^\dagger x = 1, y^\dagger x = q\}
\]  
(11)
The \(q\)-NR as well as the usual NR were first mentioned in the context of POLInSAR in [7]. The similarity of \(W_q\) with the coherence set from equation (9) is obvious. With it we can represent \(\Gamma\) as a union of \(q\)-numerical ranges over \(q\):
\[
\Gamma = \bigcup_{q \in \mathbb{C}, |q| \leq 1} W_q(\Pi)
\]  
(12)
(12) is exactly equivalent to (9) and accords to definitions of the classical general coherence in (3) and [1]. However, this set describes a filled disk around the origin of the complex plane \(W_q(\Pi) = e^{i\theta} W_q(\Pi), q = |q| e^{i\theta}\). This problem was also recognized in [1]. In order to break the phase ambiguity in coherence optimization the constraint \(\arg(\omega_1, \omega_2) = 0\) was introduced. This corresponds in the \(q\)-numerical ranges to the restriction of \(q\) to real space. Also, to remove the origin–mirror effect, \(q\) has to be positive.

In order to regard these thought we modify the \(q\)-constraint in equation (6) to \(w_1^\dagger w_2 = q, q \in [0, 1]\) and name it
Based on this restriction we can define the most general ambiguity free coherence set \( \Gamma_{2MC} \). The subscript uses the notation convention introduced in [3] and represents the use of 2 mechanisms coherence.

\[
\Gamma_{2MC} = \bigcup_{q \in [0,1]} \mathcal{W}_q(\Pi)
\]  

### 3.2 Exact One Mechanism Coherence Set

Most often, there is no need for two different scattering mechanisms. Therefore the projection vectors can be set to equal (\( \omega_1 = \omega_2 \)):

\[
\gamma_{1MC}(\omega) = \frac{\omega^\dagger\Omega_1\omega}{\sqrt{\omega^\dagger T_{11}\omega \omega^\dagger T_{22}\omega}}, \quad \omega^\dagger \omega = 1
\]  

Setting \( w = \frac{\sqrt{T_{22}\omega}}{\sqrt{\omega^\dagger T_{11}\omega} \omega^\dagger T_{22}\omega} \) and \( H = T_{22}^{-\frac{1}{2}} T_{11}^\frac{1}{2} \), we can reform equation (14) into:

\[
\gamma_{1MC}(\omega) = w^\dagger H \Pi w, \quad w^\dagger H w = 1
\]  

Please note that \( H \) is in general neither Hermitian nor positive definite. However, \( T_{11} \) and \( T_{22} \) are both positive definite after the appropriate multi–looking and have very similar eigenvectors. Because of that, \( H \) is in practice positive definite (\( \min(\Re \lambda_i) > 0 \)). Even more, \( H \) is very close to be a Hermitian matrix since the eigenvalues \( \lambda_i(i \in \{1, 2, 3\}) \) of \( H \) fulfill the constraint \( |\Re \lambda_i| \gg |\Im \lambda_i| \).

Having said this we conclude that the coherence equation (15) corresponds to the generalized inner product numerical range:

\[
\mathcal{W}_H(A) = \{ x^\dagger H A x : x \in \mathbb{C}^n, x^\dagger H x = 1 \}
\]  

With it we define the one–scattering mechanism coherence set \( \Gamma_{1MC} \) as:

\[
\Gamma_{1MC} = \{ w^\dagger H \Pi w : w \in \mathbb{C}^3, w^\dagger H w = 1 \} = \mathcal{W}_H(\Pi)
\]  

### 3.3 Approximated One Mechanism Coherence Set

Finally, often an even simpler coherence definition is used. The polarimetric matrices \( T_{ii} \) describe the same cell and often high similarity is assumed. Therefore an approximated polarimetric average matrix \( T \) is used:

\[
\gamma_{app}(\omega) = \frac{\omega^\dagger \Omega_1\omega}{\omega^\dagger T \omega}, \quad \omega^\dagger \omega = 1, \quad T = \frac{T_{11} + T_{22}}{2}
\]  

With \( \Pi = T_{22}^{-\frac{1}{2}} \Omega_{12} T_{11}^{-\frac{1}{2}} \) and \( w = \frac{\sqrt{T \omega}}{\omega^\dagger \sqrt{T \omega}} \) we get:

\[
\gamma_{app}(\omega) = w^\dagger \Pi w, \quad w^\dagger w = 1
\]  

This coherence definition corresponds to the usual NR from (10) and we are able to define an elegant coherence set:

\[
\Gamma_{app} = \mathcal{W}(\Pi)
\]  

### 4 Application of the NR

Different applications in the field of POLInSAR already exist, which take advantage of the properties of the numerical ranges in one or the other way. In this section we present some of these applications and relate them to NR characteristics. Also notably interesting features for applications of the NR will be shown. As an example application, we present an inverse analysis of the NR. In the attempt to learn more about the structure of the coherency matrix we investigate the NR of this matrix. Finally, we apply the knowledge from previous section to show that any method which uses the approximated coherence set can be modified to the use of the non–approximated coherence set, yielding more exact results.

#### 4.1 NR Properties in Applications

Directly related to the NR is the numerical radius:

\[
r(A) = \max \{|z| : z \in \mathcal{W}(A)\}
\]  

Inside of a coherence set, the numerical radius provides the maximal modulus of the coherence set. Numerical algorithms exist to compute \( r(A) \).

In [3] this property is used to optimize the coherence. In order to get second and third optimal coherences, the polarization space is reduced to subspace orthogonal to previous maximum. Therewith, up to three distinct scatterers can be separated.

The phase diversity method [7] is connected with the angular numerical range, which is a NR without any restrictions in magnitude:

\[
\mathcal{W}'(A) = \{ x^\dagger A x : x \in \mathbb{C}^n, x \neq 0 \}
\]  

In the complex plane this set has an unbounded cone-shaped form with the vertex in the origin. The aim of phase diversity is to compute the phase extremes and the corresponding polarizations. These phases define the border rays of the angular numerical range.

The NR of \( \mathcal{W}, \mathcal{W}_H, \mathcal{W}_q \) always has a convex filled shape and is always continuous, except in the trivial case of discrete one point numerical range. In [4] the numerical range shape computation is presented for the POLInSAR coherence region. This method computes a predefined number of points on the outer border of the coherence region.

#### 4.2 Coherence Set Interpretation

The properties of the NR have been widely analyzed by mathematicians. In general, [5] offers a good review on numerical ranges. At this point we want exemplarily to show possible usage of NR properties in a coherence set interpretation. As an example, we analyze a coherency matrix which is used in a parameter inversion application.
Parameter inversion techniques, that are based on the Treuhaft model [8], have a special position among the higher level PolInSAR applications. Their range extends from forest over agriculture to ice and man–built structures. In our example we survey the inversion of forest parameters from the random volume over ground model [2] at the point of contact with the NR. This application is one of the first in the field of PolInSAR. It has been widely analyzed due to its simplicity and the practical relevance in the global forest biomass estimation.

After computing the coherency matrix from model equations, the polarimetric interferometric coherency matrix \( \Pi \) has the following structure:

\[
\Pi = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{bmatrix} = \begin{bmatrix} B \\ e \end{bmatrix}, \quad a, b, c, d, e \in \mathbb{C}, \quad B \in \mathbb{C}^{2 \times 2}
\] (23)

We interpret now the properties of this matrix from the NR point of view. The NR of a complex \( 2 \times 2 \) matrix (e.g. \( B \)) has always the shape of an ellipse (possibly degenerate into a line segment or a point). The ellipse position and extension can be read out of this matrix. Another fact of NR states that the NR of a direct sum of matrices is the convex hull of the numerical ranges of these matrices. Using these properties we can give the most generalized NR shape of \( \Pi \); it is the convex hull of an ellipse (possibly degenerate) defined by \( B \) and the point \( e \) (possibly inside of the ellipse).

Knowing the constraint that the coherence set of \( \Pi \) should construct a line segment we can extract additional information. If the numerical range of a three–dimensional matrix has to be a convex polygon (e.g. a line segment), the kernel matrix has to be normal. Actually, any vertices on the outer boundary of the NR correspond to normal eigenvalues of the matrix. Furthermore, for the shape of the NR of a normal matrix to be a line segment, the vertices (i.e. eigenvalues) should be collinear in the complex plane.

With this we have two simple constraints for the modified interferometric coherency matrix \( \Pi \) to represent a line. It should be normal and the eigenvalues should be collinear in the complex plane.

4.3 Application Generalization

All of the previously presented applications work on the usual NR and with it on the approximated coherence set \( \Gamma_{\text{app}} \). Because of the shortage of paper space, we are not able to describe the mentioned methods in detail separately. At this point we want to show that the same techniques can also be applied to the non–approximated one \( \Gamma_{1,MC} \). At first we show that the generalized inner product numerical range \( W_{\Pi}(A) \) can be transformed to the usual numerical range \( W(A) \), where \( A \) is similar to \( A \) [5]. For this we assume that \( H \) is positive definite Hermitian, as mentioned in section 3.2:

\[
H = S^*S, \quad S \in \mathbb{C}^{n \times n} \text{ nonsingular (24)}
\]

Since \( S^{-1}S = I \) and with substitution \( y = Sx \) we have:

\[
W_H(A) = \{ x^*S^*SAS^{-1}Sx : x \in \mathbb{C}^n, x^*S^*Sx = 1 \}
= \{ y^*SAS^{-1}y : y \in \mathbb{C}^n, y^*y = 1 \} \quad (25)
= W\tilde{A}, \quad \tilde{A} = SAS^{-1}
\]

This means that the inner product numerical range of a matrix is equivalent to a usual NR of a similar matrix. Having this result, we can reform the one scattering mechanism coherence set from (17) to

\[
\Gamma_{1,MC} = W_H(\Pi) \approx W(\Pi), \quad \Pi = SI S^{-1} (26)
\]

In the special case that \( T_{11} \) equals \( T_{22} \), the matrix \( H \) becomes an identity matrix and \( \Pi \) will become equal to \( \Pi \) and \( \Pi \). If \( H \) does not have exactly the desired form, it need to be slightly modified. This will still yield more exact results than with the use of the approximated coherence set \( \Gamma_{\text{app}} \). Now we can use the modified matrix \( \Pi \) instead of the approximated matrix \( \Pi \) in all applications mentioned above which exploit the properties of numerical ranges.

5 Conclusions

In this paper we investigated the concept of coherence sets and their relations to numerical ranges. We characterized different types of coherence sets. Also applications of coherence sets have been reviewed and new possible applications have been emphasized.

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